

4

AD-A196 056

CCS Research Report 600

**Cone Extremal Solutions
of Multi-Payoff Games With
Cross-Constrained Strategy Sets**

A. Charnes
Z. M. Huang
J. J. Rousseau
Q. L. Wei

**CENTER FOR
CYBERNETIC
STUDIES**

**DTIC
ELECTE
S JUL 13 1988 D
E**

This document has been approved
for public release and sale; its
distribution is unlimited.

CCS Research Report 600

**Cone Extremal Solutions
of Multi-Payoff Games With
Cross-Constrained Strategy Sets**

A. Charnes
Z. M. Huang
J. J. Rousseau
Q. L. Wei

May, 1988

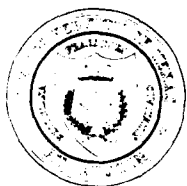
The research for this paper was partly supported by National Science Foundation Grant SES 8520806 and U.S. Army Contract DAKF-15-87-C-0110, with the Center for Cybernetic Studies at The University of Texas at Austin. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director

College of Business Administration, 5.202
The University of Texas at Austin
Austin, Texas 78712-1177
(512) 471-1821

DTIC
ELECTE
S JUL 13 1988 D
E



CENTER FOR CYBERNETIC STUDIES
THE UNIVERSITY OF TEXAS AT AUSTIN

CBA5.202 • Austin, Texas 78712-1177 • (512) 471-1821

October 24, 1988

TO: Center for Cybernetic Studies
Publications Recipients

RE: REVISIONS TO CCS 600

Attached are revisions to CCS Research Report 600, "Cone-Extremal Solutions of Multi-Payoff Games with Cross-Constrained Strategy Sets," by A. Charnes, Z. Huang, J. Rousseau and Q. Wei which was sent to you in a previous mailing.

Please replace the following pages: 10, 11, 12, 15, 16, 17, 18.
Thank you for your attention.

Attachments

Completed

Cone Extremal Solutions of Multi-Payoff Games With Cross-Constrained Strategy Sets

A. Charnes
Z. M. Huang
J. J. Rousseau
Q. L. Wei

ABSTRACT

A new solution class extending to games with non-topological product set strategies and multiple payoffs to players is developed without point-to-set mappings or quasi-variational inequalities. Some properties are developed and an illustrative example comparison to existing notions in re Pareto efficiency and Generalized Nash equilibria.

Key Words

Cone Extremal Game Solutions

Multi-Payoff Games

Interactive Strategy Sets

Quasi-variational Inequalities

Cone Extremal Solutions of Multi-Payoff Games with Cross-Constrained Strategy Sets

A. Charnes
Z. M. Huang
J. J. Rousseau
Q. L. Wei

I. Introduction

In classical game-theoretic models the strategy set is the topological product of the strategy sets of the individual players. Such models are inadequate for many purposes in economics (e.g., policy analysis for regulatory agencies) and elsewhere since they fail to take explicit account of the interactions between, and constraints on, the players' strategies, and/or the multiple objectives involved.

Extensions of classical games to include these considerations can be achieved by the new "dominance cones" method and class of solutions of Charnes, Cooper, Wei and Huang (1987c), wherein the idea of *nondominated equilibrium points* was initiated and discussed. Nondominated efficient solutions in normed vector spaces were studied by Charnes, Cooper, Wei and Huang (1987b).

In Section 2 of the present paper we introduce a new concept of "T-nondominated efficiency" for multi-payoff n-person games with **Interacting** strategy sets (see Dubey [1986] for the single payoff **non-interacting** case), where T denotes a subset of the complete set of players. Our notion of T-nondominated efficient **solutions** subsumes generalizations of both non-dominated equilibrium points (when T consists of single elements) and nondominated efficient solutions (when T is the complete set of players) as special cases. The T-nondominated efficient solutions, as we shall show, covers the spectrum from generalizations of Nash equilibria to generalizations of Pareto efficiency or optimality, without requiring point-to-set mappings for their specification.

Some essential properties of T-nondominated efficient solutions are given in Section 3 in the form of several theorems where we also examine the special case of nondominated equilibrium points. To date (see Harker [1986]), the chief generalizations of the Nash equilibrium notion for games with interactive strategy sets have been via variational inequality or quasi-variational inequality methods. Harker (1986) gives a two-player example in which only one general Nash equilibrium can be found using the variational inequality method. In contrast, in Section 4, our T-nondominated efficient solution notion enables us to determine by simple

calculation all the generalized Nash equilibrium points for this example, and we further show that none of them is Pareto-efficient.

We then determine the locus of nondominated (Pareto) efficient solutions for this example and contrast them with the nondominated (Nash) equilibrium points. In our concluding remarks we suggest a possible role for regulatory agencies in achieving some reconciliation between the two solution notions in actual cases of policy analysis.

2. Cone Convexity and T-Nondominated Efficiency

In this section we review some relevant results regarding cones and their polar cones for later use in our development and introduce a new concept of "T-Nondominated Efficiency" for multi-payoff interactive n-person games.

A set S is convex if $x_1, x_2 \in S$ implies that $\lambda x_1 + (1 - \lambda) x_2 \in S$ for all $0 \leq \lambda \leq 1$. A set S is a cone if $x \in S$ and $\lambda \geq 0$ imply that $\lambda x \in S$. S is a convex cone if S is a cone and is convex. Thus, S is a convex cone if and only if $x_1, x_2 \in S$ and $\lambda_1, \lambda_2 \geq 0$ imply that $\lambda_1 x_1 + \lambda_2 x_2 \in S$.

For an arbitrary set S in E^m , let $\bar{x} \in \bar{S}$, where \bar{S} denotes the closure of the set S . Denote the "tangency cone" of S at \bar{x} by $T(S, \bar{x})$ where $T(S, \bar{x}) = \{h \in E^m : \text{there exists a sequence } \{x^k\} \text{ and a sequence } \{\lambda^k\} \text{ such that } h = \lim_{k \rightarrow \infty} \lambda^k (x^k - \bar{x}), \text{ with } x^k \in S, \lambda^k > 0, \text{ and } \lim_{k \rightarrow \infty} x^k = \bar{x}\}$.

Further denote the (negative) polar cone of S by S^* where $S^* = \{y \in E^m : x^t y \leq 0 \text{ for all } x \in S\}$ and the superscript "t" denotes transpose. A cone Λ in E^m is said to be acute if there exists an open half-space $H = \{x \in E^m : a^t x > 0, a \neq 0\}$ such that $\bar{\Lambda} \subset H \cup \{0\}$.

The following lemma's proof may be found in [1], [2], [9], and [10].

Lemma 2.1: Let Λ and Λ_1 be cones in E^m .

- (i) if $\Lambda \subset \Lambda_1$ then $\Lambda^* \supset \Lambda_1^*$,
- (ii) $\text{Int } \Lambda^* \neq \emptyset$ if and only if Λ is acute.

(iii) When Λ is acute,

$$\text{Int } \Lambda^* = \{ y \in E^m : x^t y < 0 \text{ for all } x \in \bar{\Lambda}, x \neq 0 \} \text{ and}$$

$$\Lambda \cap (-\Lambda) = \{0\}$$

(iv) If Λ is a convex cone then $(\Lambda^*)^* = \bar{\Lambda}$

Definition 2.1: Let S be a convex set in E^m and Λ be a convex cone in E^n . A real-valued vector function $G: S \rightarrow E^n$ is called " Λ -concave on S " if

$$G(\lambda x^1 + (1-\lambda) x^2) - (\lambda G(x^1) + (1-\lambda) G(x^2)) \in \Lambda$$

$$\text{for all } x^1, x^2 \in S \text{ and } \lambda \in (0,1).$$

Definition 2.2: Let S be a convex set in E^m and Λ be a convex cone in E^n . A real-valued vector function $G: S \rightarrow E^n$ is called " Λ -quasiconcave on S " if

$$G(\lambda x^1 + (1-\lambda) x^2) - \text{Min} \{ G(x^1), G(x^2) \} \in \Lambda$$

$$\text{for all } x^1, x^2 \in S \text{ and } \lambda \in (0,1).$$

where

$$\text{Min} \{ G(x^1), G(x^2) \} = \begin{pmatrix} \min \{ g_1(x^1), g_1(x^2) \} \\ \vdots \\ \min \{ g_n(x^1), g_n(x^2) \} \end{pmatrix}$$

Lemma 2.2: Let S be a convex set in E^m and $\Lambda = E_+^n$. If $G: S \rightarrow E^n$ is Λ -quasiconcave on S , then the set

$$S_\alpha = \{ x \in S : G(x) \in \alpha + \Lambda \} \text{ is convex for every } \alpha \in E^n.$$

Proof: Let G be Λ -quasiconcave on S . For any $\alpha \in E^n$, let $x_1, x_2 \in S_\alpha$ and $\lambda \in (0,1)$. Since

$$G(\lambda x^1 + (1-\lambda) x^2) - \text{Min} \{ G(x^1), G(x^2) \} \in \Lambda$$

plus

$$G(x^i) \in \alpha + \Lambda, \quad i = 1, 2 \text{ implies } \text{Min} \{G(x^1), G(x^2)\} \in \alpha + \Lambda$$

we have

$$G(\lambda x^1 + (1-\lambda) x^2) \in \text{Min} \{G(x^1), G(x^2)\} + \Lambda \subseteq \alpha + \Lambda.$$

Thus, $\lambda x^1 + (1-\lambda) x^2 \in S_\alpha$, so that S_α is convex.

Q.E.D.

Lemma 2.3: Let S be a convex set in E^m and Λ be a convex cone in E^n . $G: S \rightarrow E^n$. If for any $\alpha \in E^n$, the set

$$S_\alpha = \{x \in S: G(x) \in \alpha + \Lambda\}$$

is convex set, then G is Λ -quasiconcave.

Proof: For any $x^1, x^2 \in S$ and $\lambda \in (0,1)$, let $\alpha = \text{Min} \{G(x^1), G(x^2)\}$. Since S_α is convex and $x^1, x^2 \in S_\alpha$, we have

$$G(\lambda x^1 + (1-\lambda) x^2) \in \alpha + \Lambda = \text{Min} \{G(x^1), G(x^2)\} + \Lambda,$$

that is,

$$G(\lambda x^1 + (1-\lambda) x^2) - \text{Min} \{G(x^1), G(x^2)\} \in \Lambda$$

Q.E.D.

Lemma 2.4: Let S be a convex set in E^n , $\Lambda \supset E_+^n$ be a convex cone, and $G: S \rightarrow E^n$ be a real-valued vector function. If G is Λ -concave on S then G is Λ -quasiconcave on S .

Proof: Let G be Λ -concave on S . Then, for any $x^1, x^2 \in S$ and $\lambda \in (0,1)$, we have

$$G(\lambda x^1 + (1-\lambda) x^2) - (\lambda G(x^1) + (1-\lambda) G(x^2)) \in \Lambda.$$

Since $\lambda G(x^1) + (1-\lambda) G(x^2) \geq \text{Min} \{G(x^1), G(x^2)\}$

we have

$$\begin{aligned} G(\lambda x^1 + (1-\lambda)x^2) &= \min \{G(x^1), G(x^2)\} \\ &\in \lambda G(x^1) + (1-\lambda)G(x^2) - \min \{G(x^1), G(x^2)\} + \Lambda \subset \Lambda \end{aligned}$$

Q.E.D.

We now give three lemmas, originally stated and proved in [1] and [2]. For further properties of cones the reader is referred to [7], [8] and [10].

Lemma 2.5: Let Λ be a closed convex cone in E^n , S be a convex set in E^m , and

$G : S \rightarrow E^n$ be differentiable in an open set which contains S . If G is Λ -concave on S , then for every $x^1, x^2 \in S$ we have

$$G(x^1) + \nabla_x G(x^1)(x^2 - x^1) \in G(x^2) + \Lambda$$

Lemma 2.6: Let S be a convex set in E^m and Λ be a convex cone in E^n with

$G : S \rightarrow E^n$. If G is Λ -concave on S , then for every $p \in (-\Lambda^*)$ we have that $p^t G$ is concave on S .

Lemma 2.7: Let S be a convex set in E^m and Λ be a closed convex cone in E^n with

$G : S \rightarrow E^n$. If, for arbitrary $p \in \Lambda^*$, $p^t G$ is concave on S , then G is $(-\Lambda)$ -concave on S .

We now state precisely what we mean by a multi-payoff, interactive n -person game, and then present the notion of T -nondominated efficiency for such games.

Definition 2.3: A multi-payoff, interactive ("cross-constrained") n -person game in normal form is given by a set of players $N = \{1, 2, \dots, n\}$; n nonempty sets $S^i \subset E^{k(i)}$, the "a priori" strategy sets of the players; a real-valued vector function $G = (g_1, \dots, g_m) : S^1 \times \dots \times S^n \rightarrow E^m$, the interactive ("cross") constraint function; n -real-valued vector functions $U^i = (u_1^i, \dots, u_l^i) : S^1 \times \dots \times S^n \rightarrow E^l$ the vector payoff functions of the players; a convex cone K in E^m , the constraint cone; a convex cone W in E^l , the dominance cone; and $X(K) = \{x = (x^1, \dots, x^n) : G(x) \in K, x^i \in S^i, i \in N\}$, the interactive strategy set. Such a game will be denoted by

$$\Gamma = \{X(K), W; u^1, \dots, u^n\}.$$

In a multi-payoff, interactive n -person game Γ , for any $x = \{x^i : i \in N\} \in S = S^1 \times \dots \times S^n$, $T \subset N$, and $y = \{y^i : i \in T\} \in \prod_{i \in T} S^i$, let $(x | y)$ denote the element of S obtained from x by replacing x^i by y^i for each $i \in T$. The concept of T -nondominated efficiency for such games, Γ , is summarized in the following definition.

Definition 2.4: A point $\bar{x} \in X(K)$ is called a

- (1) "T-nondominated efficient solution" of Γ associated with W if there does not exist any point $y \in \prod_{i \in T} S^i$, with $(\bar{x} | y) \in X(K)$ such that

$$u^i(\bar{x}) \in u^i(\bar{x} | y) + W \text{ for all } i \in T$$

$$u^j(\bar{x}) \neq u^j(\bar{x} | y) \quad \text{for some } j \in T$$

- (2) "Nondominated equilibrium point" of Γ associated with W if it is a T -nondominated efficient solution for all subsets T consisting of one element.

- (3) "Nondominated efficient solution" of Γ associated with W if it is a T -nondominated efficient solution for $T = N$.

Evidently our nondominated equilibrium point of Γ associated with W is a generalization of the Nash equilibrium point to the "games" with interacting strategy sets and vector payoff functions which were initiated and discussed in Charnes, Cooper, Wei and Huang [3]. The nondominated efficient solution of Γ associated with W is the corresponding generalization of vector or Pareto efficiency (or optimality). The T -nondominated efficient solution thus runs the gamut from Nash equilibrium generalizations to Pareto optimality generalizations, and no intervening point to set mappings are needed for their specification.

In [3] Charnes, Cooper, Wei and Huang introduced dominance cones for the study of multi-payoff, interactive n -person games, and explored the notion of a nondominated equilibrium point of Γ associated with the dominance cone W .

The focus of the present paper is on T -nondominated efficient solutions and nondominated efficient solutions and nondominated efficient solutions of Γ associated with W ; where we begin, in the following section, to examine some essential properties of such solutions. We conclude this section with the following two definitions.

Definition 2.5: $\Gamma = \{X(K), W; u^1, \dots, u^n\}$ is called a (T-W-K) -quasiconcave game if the following four conditions hold for all $i \in T$:

- (i) S^i is a convex set in $E^{k(i)}$
- (ii) $u^i(x^1, \dots, x^i, \dots, x^n)$ is (-W) -quasiconcave with respect to $\{x^j : j \in T\}$
 $\in \bigcap_{j \in T} S^j$ for fixed $\{x^k : k \in N-T\} \in \bigcap_{k \in N-T} S^k$
- (iii) $u^i(x^1, \dots, x^i, \dots, x^n)$ is continuous on $\bigcap_{j \in N} S^j$
- (iv) $G(x^1, \dots, x^i, \dots, x^n)$ is continuous and K-concave on $\bigcap_{j \in N} S^j$

Definition 2.6: $\Gamma = \{X(K), W; u^1, \dots, u^n\}$ is called a (T-W-K) - concave game if conditions (i), (iii) and (iv) of Definition 2.5 hold for all $i \in T$ and in addition, for all $i \in T$, $u^i(x^1, \dots, x^i, \dots, x^n)$ is (-W) - concave with respect to $\{x^j : j \in T\} \in \bigcap_{j \in T} S^j$ for fixed $\{x^k : k \in N-T\} \in \bigcap_{k \in N-T} S^k$.

3. Some Essential Properties of T-Nondominated Efficient Solutions.

Conditions for the existence of T-nondominated efficient solutions, and further properties of these solutions, are presented here in the form of several theorems.

Theorem 3.1: For the game $\Gamma = \{X(K), W; u^1, \dots, u^n\}$, let W be acute and S^i be compact for all $i \in T \subset N$. Then there exists at least one $\bar{x} \in X(K)$ which is a T-nondominated efficient solution of Γ associated with W .

Proof: Since W is acute, $\text{Int } W^* \neq \emptyset$. Let $p \in \text{Int } W^*$ and $U_p(x) := p^T \sum_{i \in T} u^i(x)$.

Consider the following system

$$(P) \quad \text{Max } U_p(x) \quad \text{s.t.} \quad x \in X(K).$$

Since $X(K)$ is compact, (P) has at least one optimal solution, say \bar{x} . To show that \bar{x} is a T-nondominated efficient solution of Γ associated with W, assume to the contrary that there exists $y \in \bigcup_{i \in T} S^i$ with $(\bar{x} | y) \in X(K)$ such that

$$\begin{aligned} u^i(\bar{x}) &\in u^i(\bar{x} | y) + W & \text{for all } i \in T \\ u^j(\bar{x}) &\neq u^j(\bar{x} | y) + W & \text{for some } j \in T. \end{aligned}$$

Then

$$p^t u^i(\bar{x}) \leq p^t u^i(\bar{x} | y) \quad \text{and} \quad p^t u^j(\bar{x}) < p^t u^j(\bar{x} | y).$$

Hence

$$p^t \sum_{i \in T} u^i(\bar{x}) < p^t \sum_{i \in T} u^i(\bar{x} | y)$$

That is, $u_p(\bar{x}) < u_p(\bar{x} | y)$,

which contradicts \bar{x} is an optimal solution of (P).

Q.E.D.

In the following theorem we denote the Fréchet gradient of the interactive constraint function $G(x)$ with respect to $\{x^j : j \in T\} \in \bigcup_{j \in T} S^j$ by

$$\nabla_T G(x) = \begin{pmatrix} \nabla_T g_1(x) \\ \vdots \\ \nabla_T g_m(x) \end{pmatrix}$$

where

$$\nabla_T g_i(x) = \{(\partial g_i(x) / \partial x_j), j \in T\} \text{ is a row vector.}$$

Theorem 3.2:

Let Γ be a (T-W-K) -concave game with W acute. If $\bar{x} \in X(K)$ satisfies

$$\sum_{i \in T} p_i^t \nabla_T u^i(x) + \gamma^t \nabla_T G(x) = 0 \quad (3.1)$$

$$\gamma^t G(x) = 0 \quad (3.2)$$

for some $\{p_i : p_i \in \text{Int } W^*, i \in T\}$ and $\gamma \in (-K^*)$, then \bar{x} is a T-nondominated efficient solution of Γ associated with W.

Proof: Assume to the contrary that \bar{x} is not a T-nondominated efficient solution of Γ associated with W. Then there exists $\bar{y} \in \prod_{i \in T} S^i$ with $(\bar{x} | \bar{y}) \in X(K)$ such that

$$u^i(\bar{x}) \in u^i(\bar{x} | \bar{y}) + W \text{ for all } i \in T$$

$$u^j(\bar{x}) \neq u^j(\bar{x} | \bar{y}) + W \text{ for some } j \in T,$$

and hence

$$\sum_{i \in T} p_i^t u^i(\bar{x}) < \sum_{i \in T} p_i^t u^i(\bar{x} | \bar{y}) \quad (3.3)$$

Since, for all $i \in T$, u^i and G are $(-W)$ -concave and K-concave, respectively, with respect to $\{x^i : i \in T\}$, then by Lemma 2.6 $p_i^t u^i$ and $\gamma^t G$ are concave with respect to $\{x^i : i \in T\}$.

Hence

$$p_i^t u^i(\bar{x} | \bar{y}) \leq p_i^t u^i(\bar{x}) + p_i^t \nabla_T u^i(\bar{x}) (\bar{y} - \bar{x}_T) \quad \text{for all } i \in T$$

$$\gamma^t G(\bar{x} | \bar{y}) \leq \gamma^t G(\bar{x}) + \gamma^t \nabla_T G(\bar{x}) (\bar{y} - \bar{x}_T) = \gamma^t \nabla_T G(\bar{x}) (\bar{y} - \bar{x}_T)$$

and

$$\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) (\bar{y} - \bar{x}_T) = -\gamma^t \nabla_T G(\bar{x}) (\bar{y} - \bar{x}_T)$$

$$\leq -\gamma^t G(\bar{x} | \bar{y}) \leq 0$$

where

$$\bar{x}_T = \{\bar{x}^i : i \in T\}.$$

Then

$$\begin{aligned} \sum_{i \in T} p_i u^i(\bar{x} | \bar{y}_T) &\leq \sum_{i \in T} p_i u^i(\bar{x}) + \sum_{i \in T} p_i \nabla_T u^i(\bar{x}) (\bar{y} - \bar{x}_T) \\ &\leq \sum_{i \in T} p_i^1 u^i(\bar{x}) \end{aligned}$$

which contradicts (3.3).

Q.E.D.

Assumption (A): For any $\bar{x} \in X(K)$, $\bar{y} \in \prod_{i \in T} S^i$ and $\hat{y} \in \prod_{i \in T} S^i$, let

$$U_c^i = \min \{u^i(\bar{x} | \hat{y}), u^i(\bar{x} | \bar{y})\}, \quad \hat{U}_c = \min_{j \in T} U^j(\bar{x} | \hat{y}), \quad \bar{U}_c = \min_{j \in T} U^j(\bar{x} | \bar{y})$$

then $\lambda (\hat{U}_c - U_c^i) + (1-\lambda) (\bar{U}_c - U_c^i) \in W$ for each $\lambda \in [0, 1]$.

Lemma 3.1 : Let Γ be a (T-W-K) -quasiconcave game and $W \supset E^f$ be acute. If $\bar{x} \in X(K)$ is a T-nondominated efficient solution of Γ associated with W and Assumption (A) holds, then there exist $p_i \in W^*$, $i \in T$, not all zero, such that

$$\sum_{i \in T} p_i u^i(\bar{x}) \geq \sum_{i \in T} p_i u^i(\bar{x} | y) \quad \text{for all } y \in \prod_{i \in T} S^i \text{ with } (\bar{x} | y) \in X(K)$$

Proof: Consider

$$\Lambda = \left\{ \{z^i : i \in T\} : \text{there exists some } \bar{y} \in \prod_{i \in T} S^i \text{ with } (\bar{x} | \bar{y}) \in X(K) \right.$$

$$\left. \text{such that } z^i - \min_{j \in T} u^j(\bar{x} | \bar{y}) + u^i(\bar{x}) \in W. \right.$$

$$\left. \text{for all } i \in T \text{ and } z^k - \min_{j \in T} u^j(\bar{x} | \bar{y}) + u^k(\bar{x}) \neq 0. \right.$$

$$\left. \text{for at least one } k \in T \right\}.$$

First we need to show that Λ is a convex set. Accordingly, let $\{\hat{z}^i : i \in T\}$ and $\{\hat{z}^i : i \in T\}$

lie in Λ . That is, there exist $\hat{\bar{y}}$ and $\bar{\bar{y}}$ in $\prod_{i \in T} S^i$ with

$(\bar{x} | \hat{y}^0) \in X(K)$ and $(\bar{x} | \hat{y}) \in X(K)$ such that

$$\hat{z}^i = \min_{j \in T} u^j(\bar{x} | \hat{y}^0) + u^i(\bar{x}) \in W \quad \text{for all } i \in T$$

$$\hat{z}^{k_0} = \min_{j \in T} u^j(\bar{x} | \hat{y}^0) + u^{k_0}(\bar{x}) \neq 0 \quad \text{for some } k_0 \in T$$

and

$$\hat{z}^i = \min_{j \in T} u^j(\bar{x} | \hat{y}) + u^i(\bar{x}) \in W \quad \text{for all } i \in T$$

$$\hat{z}^{k_1} = \min_{j \in T} u^j(\bar{x} | \hat{y}) + u^{k_1}(\bar{x}) \neq 0 \quad \text{for some } k_1 \in T$$

For any $0 < \lambda < 1$ we then obtain

$$\lambda \hat{z}^i + (1-\lambda) \hat{z}^i = \left[\lambda \min_{j \in T} u^j(\bar{x} | \hat{y}^0) + (1-\lambda) \min_{j \in T} u^j(\bar{x} | \hat{y}) \right] + u^i(\bar{x}) \in W \quad \text{for all } i \in T.$$

Since Γ is a (T-W-K)-quasiconcave game we have

$$\min \left\{ u^i(\bar{x} | \hat{y}^0), u^i(\bar{x} | \hat{y}) \right\} - u^i \left[\lambda (\bar{x} | \hat{y}^0) + (1-\lambda) (\bar{x} | \hat{y}) \right] \in W.$$

$$\text{Let } u_c^i = \min \left\{ u^i(\bar{x} | \hat{y}^0), u^i(\bar{x} | \hat{y}) \right\}$$

$$u_c^0 = \min_{j \in T} u^j(\bar{x} | \hat{y}^0)$$

$$\hat{u}_c = \min_{j \in T} u^j(\bar{x} | \hat{y}).$$

Hence

$$\lambda u_c^0 + (1-\lambda) \hat{u}_c = \left[\lambda (u_c^0 - u_c^0) + (1-\lambda) (\hat{u}_c - u_c^0) \right]$$

$$- u \left[\lambda (\bar{x} | \hat{y}^0) + (1-\lambda) (\bar{x} | \hat{y}) \right] + u^i(\bar{x}) \in W \quad \text{for all } i \in T.$$

and therefore by Assumption (A),

$$\lambda \hat{z}^i + (1-\lambda) \hat{z}^i - u^i \left[\lambda (\bar{x}|y^i) + (1-\lambda) (\bar{x}|\hat{y}) \right] + u^i(\bar{x}) \in W \text{ for all } i \in T.$$

By Lemma 2.1 we have

$$\lambda z^i + (1-\lambda) \hat{z}^i - u^i \left[\lambda (\bar{x}|y^i) + (1-\lambda) (\bar{x}|\hat{y}) \right] + u^i(\bar{x}) \neq 0$$

that is,

$$\lambda \{z^i : i \in T\} + (1-\lambda) \{\hat{z}^i : i \in T\} \in \Lambda$$

Hence Λ is a convex set.

Clearly, $0 \notin \Lambda$, so by the convex separation theorem there exist $p_i, i \in T$, not all zero, such that

$$\sum_{i \in T} p_i z^i \leq 0 \text{ for all } \{z^i : i \in T\} \in \Lambda$$

For any $y \in \bigcup_{i \in T} S^i$ with $(\bar{x}|y) \in X(K)$, $w^i \in W, i \in T$, not all zero, and $\lambda > 0$, set

$$z^i = u^i(\bar{x}|y) - u^i(\bar{x}) + \lambda w^i.$$

Then

$$\{z^i : i \in T\} \in \Lambda \text{ and}$$

$$\sum_{i \in T} p_i u^i(\bar{x}|y) - \sum_{i \in T} p_i u^i(\bar{x}) + \lambda \sum_{i \in T} p_i w^i \leq 0.$$

Hence $p_i \in W^*, i \in T$.

Letting $\lambda \rightarrow 0^+$, we then have

$$\sum_{i \in T} p_i u^i(\bar{x}|y) \leq \sum_{i \in T} p_i u^i(\bar{x})$$

for all $y \in \prod_{i \in T} S^i$ with $(\bar{x}|y) \in X(K)$.

Q.E.D.

The following lemma, whose proof parallels that of Lemma 3.1, is also useful in more general contexts.

Lemma 3.2: Let Γ be a $(T-W-K)$ -concave game and W be acute. If $\bar{x} \in X(K)$ is a T -nondominated efficient solution of Γ associated with W , then there exist $p_i \in W^*$, $i \in T$, not all zero, such that

$$\sum_{i \in T} p_i^1 u^i(\bar{x}) \geq \sum_{i \in T} p_i^1 u^i(\bar{x}|y)$$

for all $y \in \prod_{i \in T} S^i$ with $(\bar{x}|y) \in X(K)$.

Now for fixed $\bar{x} \in X(K)$ let

$$D_T(\bar{x}) = \left\{ y \in \prod_{i \in T} S^i : (\bar{x}|y) \in X(K) \right\}.$$

Theorem 3.3: Let $\bar{x} \in X(K)$. If there exists $\{p_i : i \in T\}$ with $p_i \in W^*$ such that

$$\sum_{i \in T} p_i^1 u^i(\bar{x}) \geq \sum_{i \in T} p_i^1 u^i(\bar{x}|y) \text{ for all } y \in \prod_{i \in T} S^i \text{ with } (\bar{x}|y) \in X(K)$$

then

$$\left[\sum_{i \in T} p_i^1 \nabla_T u^i(\bar{x}) \right]^t \in T^*(D_T(\bar{x}), \bar{x}_T)$$

Proof: For any $h \in T(D_T(\bar{x}), \bar{x}_T)$, we need to show only that

$$\left(\sum_{i \in T} p_i^1 \nabla_T u^i(\bar{x}) \right) h \leq 0.$$

Thus, let $\{y^k\} \subset D_T(\bar{x})$ with $\lim_{k \rightarrow \infty} y^k = \bar{x}_T$, and $\lambda_k > 0$ with $\lim_{k \rightarrow \infty} \lambda_k = 0$ such that

$$h = \lim_{k \rightarrow \infty} \lambda_k (y^k - \bar{x}_T)$$

$$\text{Since } \sum_{i \in T} p_i^t u^i(\bar{x} | y^k) = \sum_{i \in T} p_i^t u^i(\bar{x}) + \sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) (y^k - \bar{x}_T) + o(\|y^k - \bar{x}_T\|)$$

$$\text{And } \sum_{i \in T} p_i^t u^i(\bar{x} | y^k) \leq \sum_{i \in T} p_i^t u^i(\bar{x})$$

we have

$$\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) (y^k - \bar{x}_T) + o(\|y^k - \bar{x}_T\|) \leq 0$$

and therefore

$$\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) \lambda_k (y^k - \bar{x}_T) + \lambda_k \cdot o(\|y^k - \bar{x}_T\|) \leq 0.$$

Letting $k \rightarrow \infty$, we then obtain

$$\left(\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) \right) h \leq 0.$$

Q.E.D.

Assuming $\bar{x} \in X(K)$, let

$$C_T(\bar{x}) = \left\{ \left(\nabla_T G(\bar{x}) \right)^t \gamma : \gamma \in -K^* \text{ with } \gamma^t G(\bar{x}) = 0 \right\}$$

Lemma 3.3:^[2] Let $\bar{x} \in X(K)$ and $G(x)$ be Fréchet differentiable at \bar{x} with respect to $\{x^i : i \in T\}$. Then

$$T(D_T(\bar{x}), \bar{x}_T) \subset (-C_T^*(\bar{x}))$$

By Lemma 2.1: $T^*(D_T(\bar{x}), \bar{x}_T) \supset (-C_T(\bar{x}))$. In the general case, $C_T(\bar{x})$ is not a closed set, but if we let $K = E_+^m$ and $T = N$, then we have

$$C_T(\bar{x}) = \left\{ \sum_{i \in T} \gamma_i \nabla_T g_i(\bar{x}) : \gamma_i \geq 0, i \in I \right\}$$

where $I = \{i : g_i(\bar{x}) = 0, 1 \leq i \leq m\}$

We know that $C_T(\bar{x})$ is a closed set (a simple proof is given in [6]). Thus we can make the following definition.

Definition 3.1: A point $\bar{x} \in X(K)$ is said to be a "T-regular point" of the constraint set $X(K)$ if $T^*(D_T(\bar{x}), \bar{x}) \subset (-C_T(\bar{x}))$.

Theorem 3.4: Let Γ be a (T-W-K) -quasiconcave game and $W \supset E^L$ be acute. If $\bar{x} \in X(K)$ is a T-nondominated efficient solution of Γ associated with W and a T-regular point of $X(K)$, and Assumption (A) holds, then there exist $p_i \in W^*, i \in T$, not all zero, and $\gamma \in (-K^*)$ such that

$$\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) + \gamma^t \nabla_T G(\bar{x}) = 0$$

$$\gamma^t G(\bar{x}) = 0$$

Proof: By Lemma 3.1 there exist $p_i \in W^*, i \in T$, not all zero, such that

$$\sum_{i \in T} p_i^t u^i(\bar{x}) \geq \sum_{i \in T} p_i^t u^i(\bar{x}|y) \quad \text{for all } y \in \bigcap_{i \in T} S^i \text{ with}$$

$(\bar{x}|y) \in X(K)$.

By Theorem 3.3 we have

$$\left[\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) \right]^t \in T^*(D_T(\bar{x}), \bar{x}_T).$$

Since \bar{x} is a T-regular point, we know that

$$\left[\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) \right]^t \in (-C_T(\bar{x}))$$

Then there exists $\gamma \in (-K^*)$ with $\gamma^t G(\bar{x}) = 0$ such that

$$\left[\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) \right]^t = -(\nabla_T G(\bar{x}))^t \gamma,$$

that is,

$$\sum_{i \in T} p_i^t \nabla_T u^i(\bar{x}) + \gamma^t \nabla_T G(\bar{x}) = 0$$

$$\gamma^t G(\bar{x}) = 0$$

Q.E.D.

Evidently by Lemma 3.2, Theorem 3.4 also holds if Γ is a (T-W-K) -concave game with any acute W and we do not need that Assumption (A) holds.

The following two useful theorems can be derived now directly from Theorem 3.2 and Theorem 3.4.

Theorem 3.5: For all subsets T consisting of a single element, let Γ be a (T-W-K) -concave game with W acute. If $\bar{x} \in X(K)$ satisfies (3.1) and (3.2) then \bar{x} is a non-dominated equilibrium point of Γ associated with W.

Theorem 3.6: For all subsets T consisting of a single element, let Γ be a (T-W-K) -quasiconcave game with acute $W \supset E^L$. If $\bar{x} \in X(K)$ is a nondominated equilibrium point of Γ associated with W and a T-regular point, and Assumption (A) holds, then for any fixed T consisting of one element, there exists nonzero $p \in W^*$ and $\gamma \in (-K^*)$ such that

$$p^t \nabla_T u^i(\bar{x}) + \gamma^t \nabla_T G(\bar{x}) = 0, \text{ for } i \in T$$

$$\gamma^t G(\bar{x}) = 0$$

If we use that Γ is a (T-W-K) -concave game, then the result is also true without the Assumption (A).

4. An Illustrative Example

We now turn to the example given by Harker [5], the two-person game depicted in Figure 1. Each player chooses a number x_i between 0 and 10 such that the sum of these numbers is less than or equal to 15. Harker's utility functions and constraint function are defined in our terms by:

$$\begin{aligned} u^1(x_1, x_2) &= 34x_1 - x_1^2 - \frac{8}{3}x_1x_2 \\ u^2(x_1, x_2) &= 24.25x_2 - x_2^2 - \frac{5}{4}x_1x_2 \\ g(x_1, x_2) &= 15 - x_1 - x_2. \end{aligned}$$

$$\text{Here, } W = E^1, K = E^1_+, S^1 = \{x_1: 0 \leq x_1 \leq 10\},$$

$$S^2 = \{x_2: 0 \leq x_2 \leq 10\}, \text{ and}$$

$$X(K) = \{(x_1, x_2) : x_1 \in S^1, x_2 \in S^2, g(x_1, x_2) \geq 0\}.$$

Since $g(x_1, x_2)$ is linear, for all subsets T consisting of a single element and any $(x_1, x_2) \in X(K)$, (x_1, x_2) is a T -regular point. Further, it is easy to show that if T consists of one element, $\Gamma = \{X(K), W; u^1, u^2\}$ is a (T-W-K) -concave game (otherwise it is quasi-concave). Hence, by Theorems 3.5 and 3.6, the nondominated equilibrium points of Γ are points which satisfy (3.1) and (3.2), and vice-versa.

From (3.1) and (3.2), we need to find points (x_1, x_2) in $X(K)$ that satisfy the following equations:

$$34 - 2x_1 - \frac{8}{3}x_2 - \gamma_1 = 0 \tag{4.1}$$

$$24.25 - 2x_2 - \frac{5}{4}x_1 - \gamma_2 = 0 \tag{4.2}$$

$$\gamma_1(15 - x_1 - x_2) = 0 \quad (4.3)$$

$$\gamma_2(15 - x_1 - x_2) = 0 \quad (4.4)$$

$$\gamma_1, \gamma_2 \geq 0.$$

If $x_1 + x_2 < 15$, then by (4.3) and (4.4), we must have $\gamma_1 = \gamma_2 = 0$. (4.1) and (4.2) then reduce to

$$34 - 2x_1 - \frac{8}{3}x_2 = 0$$

$$24.25 - 2x_2 - \frac{5}{4}x_1 = 0$$

with solution $(x_1, x_2) = (5, 9)$.

If $x_1 + x_2 = 15$, we may substitute $15 - x_1$ for x_2 in (4.1) and (4.2), and using $\gamma_1, \gamma_2 \geq 0$ we obtain $x_1 \geq 9$ and $x_1 \geq 23/3$. Since $x_1 \geq 9$ implies $x_1 \geq 23/3$, the interval $[(9, 6), (10, 5)]$ satisfies (4.1) through (4.4).

Therefore, by Theorems 3.5 and 3.6, the set of non-dominated equilibrium points of Γ (Nash equilibrium points in this case) is comprised of the point $(5, 9)$ and the interval

$[(9, 6), (10, 5)]$, as shown in Figure 1.

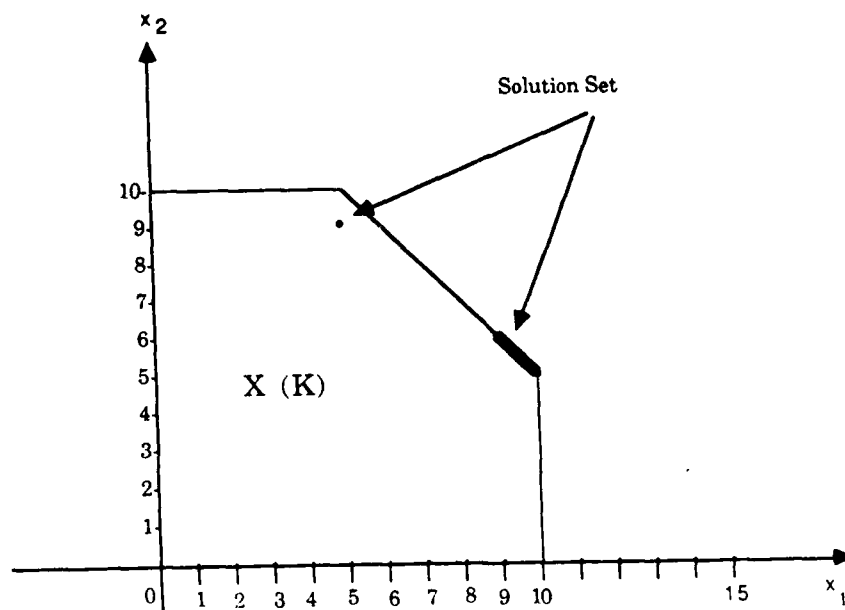


Figure 1

As is well known (see [4]), Nash equilibria in general are **not** Pareto efficient solutions. We now show that **none** of the Nash equilibrium points in the above example are Pareto-efficient and then proceed to determine the locus of Pareto-efficient points.

Take $T = N = \{1, 2\}$. Since

$$\nabla^2 u^1 = \begin{pmatrix} -2, & -\frac{8}{3} \\ -\frac{8}{3}, & 0 \end{pmatrix} \text{ and } \nabla^2 u^2 = \begin{pmatrix} 0, & -\frac{5}{4} \\ -\frac{5}{4}, & -2 \end{pmatrix}$$

are neither positive semi-definite nor negative semi-definite matrices, u^1 and u^2 are neither convex nor concave functions.

We first show that u^1 and u^2 are quasiconcave on $S^1 \times S^2$. Note that $u^i(x_1, x_2) \geq 0$ on $S^1 \times S^2$ for $i = 1, 2$. Hence:

$$S_\alpha^1 = \left\{ (x_1, x_2) \in S^1 \times S^2 : u^1(x_1, x_2) \geq \alpha \right\}$$

and

$$S_\alpha^2 = \left\{ (x_1, x_2) \in S^1 \times S^2 : u^2(x_1, x_2) \geq \alpha \right\}$$

are convex sets for all $\alpha \leq 0$.

For all $\alpha > 0$ we have

$$\begin{aligned} S_\alpha^1 &= \left\{ (x_1, x_2) \in S^1 \times S^2 : u^1(x_1, x_2) \geq \alpha \right\} \\ &= \left\{ (x_1, x_2) \in S^1 \times S^2 : x_2 \leq \frac{3}{8} \left(34 - x_1 - \frac{\alpha}{x_1} \right) \right\} \end{aligned}$$

and

$$\begin{aligned} S_\alpha^2 &= \left\{ (x_1, x_2) \in S^1 \times S^2 : u^2(x_1, x_2) \geq \alpha \right\} \\ &= \left\{ (x_1, x_2) \in S^1 \times S^2 : x_1 \leq \frac{4}{5} \left(24.25 - x_2 - \frac{\alpha}{x_2} \right) \right\}. \end{aligned}$$

Clearly, S_α^1 and S_α^2 are convex sets, so by Lemma 2.3 u^1 and u^2 are quasiconcave on $S^1 \times S^2$.

Then by Theorem 3.4, if $(x_1, x_2) \in X(K)$ is a Pareto-efficient solution then there exist $(\lambda_1, \lambda_2) \geq 0$ with $(\lambda_1, \lambda_2) \neq (0, 0)$ and $\gamma \geq 0$ such that

$$\begin{aligned}\lambda_1 \nabla_T u^1(x_1, x_2) + \lambda_2 \nabla_T u^2(x_1, x_2) + \gamma \nabla_T g(x_1, x_2) &= 0 \\ \gamma g(x_1, x_2) &= 0,\end{aligned}$$

that is

$$\lambda_1 \left(34 - 2x_1 - \frac{8}{3} x_2 \right) + \lambda_2 \left(-\frac{5}{4} x_2 \right) - \gamma = 0 \quad (4.5)$$

$$\lambda_1 \left(-\frac{8}{3} x_1 \right) + \lambda_2 \left(24.25 - 2x_2 - \frac{5}{4} x_1 \right) - \gamma = 0 \quad (4.6)$$

$$\gamma(15 - x_1 - x_2) = 0 \quad (4.7)$$

(i) For the point $(x_1, x_2) = (5, 9)$, substitution in (4.5) - (4.7) leads to

$\gamma = \lambda_1 = \lambda_2 = 0$. Therefore, the point (5, 9) is not Pareto-efficient.

(ii) Consider the interval $I = [(9, 6), (10, 5)]$. Substituting $15 - x_1$ for x_2 in (4.5) and (4.6) we obtain

$$-6\lambda_1 + \frac{2}{3}\lambda_1 x_1 - \frac{75}{4}\lambda_2 + \frac{5}{4}\lambda_2 x_1 = \gamma \quad (4.8)$$

$$-\frac{8}{3}\lambda_1 x_1 - \frac{23}{4}\lambda_2 + \frac{3}{4}\lambda_2 x_1 = \gamma \quad (4.9)$$

from which we derive

$$\lambda_2 = \frac{36 - 20x_1}{3x_1 - 78} \lambda_1. \quad (4.10)$$

If $\lambda_1 = 0$, then $\lambda_2 = \gamma = 0$. If $\lambda_2 = 0$, then for $9 \leq x_1 \leq 10$ we also have

$\lambda_1 = \gamma = 0$. In neither case will any point in the interval I be Pareto-efficient.

Substituting for λ_2 from (4.10) in (4.8) we obtain

$$\gamma_2 = \frac{-23x_1^2 + 350x_1 - 207}{3x_1 - 78} \lambda_1 \quad (4.11)$$

which is less than zero for any $9 \leq x_1 \leq 10$ and $\lambda_1 > 0$. Thus, no equilibrium point in the interval I is Pareto-efficient.

The Pareto-efficient points for this example constitute in part the intersection of the constraint set with the locus of points of tangency between the two players' contour of constant utility. This is the "contract curve" in economics parlance; any departure from this curve cannot improve the lot of both players and will leave at least one player worse off. There are also other Pareto-efficient points stemming from the fact that the sum of 2 quasi-concave functions need not be quasi-concave.

The locus of points of tangency is given by $\nabla u^1 + \lambda \nabla u^2 = 0$ for $\lambda \geq 0$.

Since $\nabla u^1 = (34 - 2x_1 - \frac{8}{3}x_2, -\frac{8}{3}x_1)$ and $\nabla u^2 = (-\frac{5}{4}x_2, 24.25 - 2x_2 - \frac{5}{4}x_1)$, we then have

$$34 - 2x_1 - \frac{8}{3}x_2 - \frac{5}{4}\lambda x_2 = 0 \quad (4.12)$$

$$24.25\lambda - 2\lambda x_2 - \frac{5}{4}\lambda x_1 - \frac{8}{3}x_1 = 0 \quad (4.13)$$

Eliminating λ and collecting terms we obtain

$$4947 - 546x_1 + 15x_1^2 + 24x_1x_2 - 796x_2 + 32x_2^2 = 0 \quad (4.14)$$

Hence the set of Pareto-efficient points is given by

$$P = \left\{ (x_1, x_2) \in S^1 \times S^2 : 4947 - 546x_1 + 15x_1^2 + 24x_1x_2 - 796x_2 \right. \quad (4.15)$$

$$\left. + 32x_2^2 = 0, x_1 + x_2 \leq 15 \right\} \cup \{ (x_1, x_2) : 0 \leq x_1 \leq 0.63, x_2 = 10 \}$$

$$\cup \{ (x_1, x_1) : x_1 = 10, 0 \leq x_2 \leq 2 \}$$

Figure 2 depicts (part of) two representative contours of constant utility for each player: $(u^1, u^2) = (41, 91)$, with point of tangency $(x^1, x^2) = (3, 6.5)$, and $(u^1, u^2) = (77.88, 65.27)$ with point of tangency $(x^1, x^2) = (5, 5.03)$, to two decimal places. The locus of non-dominated (Pareto) efficient solutions extends from $(x^1, x^2) = (0, 10)$, to $(x^1, x^2) = (10, 0)$, with corresponding utility values ranging from $(u^1, u^2) = (0, 142.5)$ to $(u^1, u^2) = (240, 0)$.

The non-dominated equilibrium point $(x^1, x^2) = (5, 9)$ corresponds to utility values $(u^1, u^2) = (25, 81)$, while the interval $[(9, 6), (10, 5)]$ corresponds to the utility interval $[(81, 42), (106.67, 33.75)]$.

5. Concluding Remarks

We have presented a new concept of "T-non-dominated efficiency" for multi-payoff n-person games with interacting strategy sets. Our notion of T-non-dominated efficient solutions subsumes generalizations of both non-dominated equilibrium points and non-dominated efficient solutions, and thus covers the spectrum from generalizations of Nash equilibria to generalizations of Pareto efficiency or optimality for intersecting or cross-constrained strategy sets.

Several theorems have presented essential properties of T-non-dominated solutions and non-dominated equilibrium points, and we have illustrated their application by means of a two-player example in which only one generalized Nash equilibrium point can be found by variational inequality methods. In contrast, our new solution concept and theory enables us to determine easily all the generalized Nash equilibrium points for this example, none of which is Pareto efficient, as well as the locus of Pareto-efficient solutions.

Because of non-uniqueness of the generalized Nash equilibrium points any one chosen is not automatically the "best" from the point of view of the competing parties or "society". Moreover, Nash equilibria in general are not Pareto efficient, and hence will not necessarily be optimal or desirable from the point of view of society. For many purposes in economics Pareto-efficient solutions are needed.

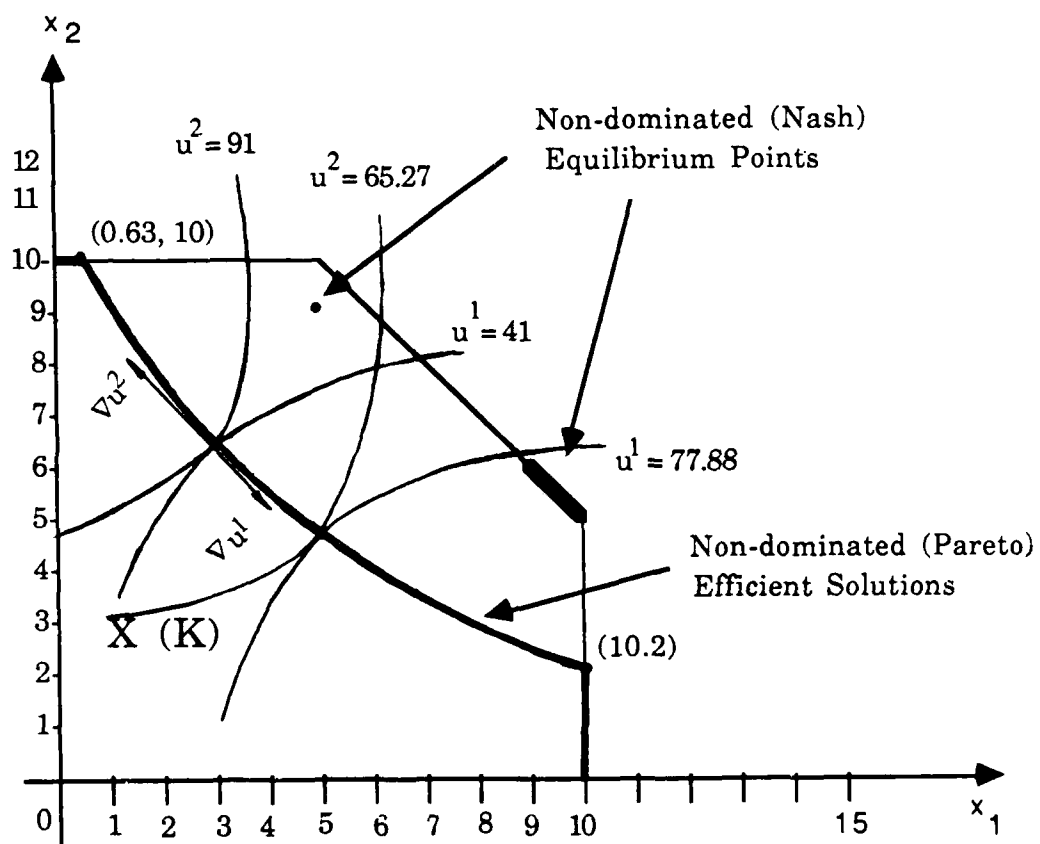


Figure 2

For example, while seeking to preserve competition, the overall good of both consumers and suppliers requires a balance or regulation of competition which amounts effectively to a cooperation imposed by the regulatory agency. Pareto-efficient solutions thus embody notions of "collective" or "societal" stability, in which no improvement can be provided to all parties simultaneously and in which a departure by any party from a Pareto-efficient solution necessarily results in some other party becoming worse off.

If (generalized) Nash equilibria are indeed the result of (unregulated) competitive action, and such equilibrium solutions do not intersect with Pareto-efficient or other society-desired solutions, can regulation achieve this intersection?

Or is it possible (and desirable) only to achieve a weaker T-nondominated solution? Evidently our new notion provides the important existence/flexibility evaluations of what is possible.

References

- [1] Charnes, A., Cooper, W. W., Wei, Q. L. and Huang, Z. M., "Cone-Ratio Data Envelopment Analysis and Multi-objective Programming," Research Report CCS 559, Center for Cybernetic Studies, The University of Texas at Austin, January, 1987a.
- [2] Charnes, A., Cooper, W. W., Wei, Q. L. and Huang, Z. M., "Fundamental Theorems of Non-dominated Solutions Associated with Cones in Normed Linear Spaces," Research Report CCS 575 Center for Cybernetic Studies, The University of Texas at Austin, June, 1987b
- [3] Charnes, A., Cooper, W. W., Wei, Q. L. and Huang, Z. M., "Multi-Payoff Constrained N-Person Games Over Cones," Research Report CCS 583 Center for Cybernetic Studies, The University of Texas at Austin, June, 1987c
- [4] Dubey, P., "Inefficiency of Nash Equilibria," Mathematics of Operations Research, Vol. 11 No. 1, February, 1986, p. 1-8.
- [5] Harker, P. T., "Generalized Nash Games and Quasi-Variational Inequalities," Working Paper 86-05-03, Department of Decision Science, University of Pennsylvania, Philadelphia, Pennsylvania, May, 1986.
- [6] Huang, Z. M., "A Note About Farkas Theorem," Journal of the Beijing Institute Aeronautics and Astronautics, 4 (1983).
- [7] Huang, Z. M., "The Second Order Conditions of Non-dominated Solutions for Generalized Multi-objective Mathematical Programming," Journal of Systems Science and Mathematical Science, 5(1985)3.
- [8] Rockafellar, R. T., Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.
- [9] Storer, J. and Witzgall, C., "Convexity and Optimization in Finite Dimensions 1", Springer-Verlag, New York, New York, 1970.
- [10] Yu, P. L., "Cone Convexity, Cone Extreme Points and Non-dominated Solutions in Decision Problems with Multi-objectives," Journal of Optimization Theory and Applications 14(1974)3.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CCS 600	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CONE EXTREMAL SOLUTIONS OF MULTI-PAYOFF GAMES WITH CROSS-CONSTRAINED STRATEGY SETS		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER CCS 600
7. AUTHOR(s) A. Charnes, Z.M. Huang, J.J. Rousseau, Q.L. Wei		8. CONTRACT OR GRANT NUMBER(s) NSF SES 8520806 ARMY DAKF15-87-C-0110
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Cybernetic Studies, UT Austin Austin, Texas 78712		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS USAREC Ft. Sheridan, IL 60037		12. REPORT DATE May 1988
		13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) This document has been approved for public release and sale; its distribution is unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Cone Extremal Game Solutions, Multi-Payoff Games, Interactive Strategy Sets, Quasi-variational Inequalities.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A new solution class extending to games with non-topological product set strategies and multiple payoffs to players is developed without point-to-set mappings or quasi-variational inequalities. Some properties are developed and an illustrative example comparison to existing notions re Pareto efficiency and Generalized Nash equilibria.		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 68 IS OBSOLETE
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)